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On relative intensity measurements with a two-directional focusing β-spectrometer with energy-depending window curve

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With 6 figures in the text

A. Introduction

The two-directional focusing β-spectrometer first proposed by Siegbahn and Svartholm [1] has become extensively used in β-spectroscopy for the determinations of energies and intensities of β-particles. The accuracy which can be obtained in such determinations depends, of course, upon the more detailed conditions of the measurements and the properties of the complete spectrometer set-up. In the following work we shall treat a special problem connected with intensity determinations, which is present in the case when pole shoes of iron are used to produce the desired magnetic field. Because the permeability of the iron varies with the field strength, the shape of the field is not independent of the magnetizing current. Due to this variation in the field shape, the window curve of the spectrometer will depend upon the energy of the focused particles. With the field strengths generally used in β-spectroscopy, the change of the permeability is most pronounced at low field strengths, which means that the effect is most disturbing in spectrometers of large dimensions and at low energies. Some uncertainty seems to have been present concerning the reliability of intensity determinations under such conditions. Especially when investigating more complex decays, the demand for good intensity measurements is, of course, essential. Recent improvements of the calculation of conversion coefficients have also emphasized the importance of careful intensity measurements. It may therefore be worthwhile to investigate how a variation of the field form affects the intensity values obtained. It will be shown below that intensity determinations are rather insensitive to changes in the window curve both when comparing intensities of conversion lines and when measuring the shape of a continuous β-spectrum. A simple method is also given to reduce the weak energy dependence present. Finally, the results obtained are experimentally tested.

B. Some definitions

Let the radioactive sample, which we shall, for simplicity, assume to be a point source, radiate $I(p)\,dp$ electrons per unit solid angle and unit time. We call $I(p)$ the true intensity at momentum p.

Suppose that we measure the counting rate N counts per unit time as a function of the field strength H at some suitable references point of the magnetic field of the spectrometer. We call $N(H)$ the experimental intensity at H.

The statement that the window curve is not constant means that it is not possible to write for the contribution $dN(H)$ to $N(H)$

$$dN(H) = \text{const}\, I(p)\, dp\, W\left(\frac{H}{p}\right),$$

where the window curve W is a function of H/p only.

Let H^* be the value of H at a well-defined point of the window curve and p the corresponding momentum of the electrons. The relation between p and H^* is expressed by a calibration function $k(H^*)$ such that $p = k(H^*)H^*$.

Our task is to find some useful relation between the experimental intensity $N(H)$ and the true intensity $I(p)$ and investigate how this relation varies with the field shape.

The two-directional focusing spectrometer is mostly used for measurements at high and moderate resolution, which means that the base width of a line does not exceed 2–3 %. We limit the following considerations to these cases and assume that it is sufficient to know intensities with an accuracy of about 1 %.

C. Two expressions for the relation between $N(H)$ and $I(p)$

We shall consider the basic spectrometer geometry shown in fig. 1. The point source is situated on the central line $r = r_0$. $z = 0$ in the source plane S. The orbits are radially defined by an opening in the plane R and axially by an opening in the plane A. The detector slit is placed in the plane D and is radially extended between $r = r_0 - d$ and $r = r_0 + d$. The slit is supposed to be sufficiently high to prevent particles from being lost when axial defocusing is present. We shall use the following entities to describe the path for electrons of momentum p.

γ the angle between the path and the circle $r = r_0$ in the starting point.
θ the angle between the plane $z = 0$ and the tangent plane to the path and the circle $r = r_0$ in the starting point.
φ the angular coordinate.
r, z and H as introduced above.

The field shape is described by a number of field parameters $\alpha_1, \alpha_2, \alpha_3, \ldots$ which evidently must be functions of H. We shall assume in the following that every effect due to the change of the field form over an interval of H which is contained under a line profile can be neglected. The justification of such a "static approximation" will be discussed later.

Let the radial dispersion be $v_r(\gamma, \theta, r, \varphi, \alpha)$ where the dependence of v_r upon the parameters $\alpha_1, \alpha_2, \ldots$ has been abbreviated by writing only α. In the detector plane $\varphi = \pi\sqrt{2}$ we have

$$dr = v_r(\gamma, \theta, r, \pi\sqrt{2}, \alpha)\frac{dH}{p}. \tag{1}$$

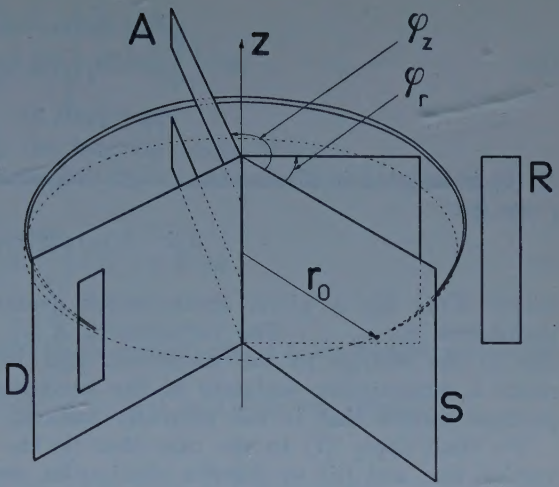

Fig. 1. Schematic drawing showing the spectrometer geometry considered in the present work.

Consider the bundle in fig. 1. The number of particles which pass a cross-section of it per unit time is

$$I(p)\,d\,p\,\sin\gamma\,d\gamma\,d\,\theta. \tag{2}$$

The bundle can pass through the dectector slit when H has values between H_1 and H_2, where according to (1) we have

$$H_2-H_1=\int\limits_{r_0-d}^{r_0+d}\frac{p\,d\,r}{v_r\,(\gamma,\,\theta,\,r,\,\pi\sqrt{2},\,\alpha)}. \tag{3}$$

For the area d^3A, which the bundle gives in the $N-H$-plane, we get from (2) and (3)

$$d^3A=I\,(p)\,d\,p\,\sin\gamma\,d\gamma\,d\,\theta\int\limits_{r_0-d}^{r_0+d}\frac{p\,d\,r}{v_r\,(\gamma,\,\theta,\,r,\,\pi\sqrt{2},\,\alpha)} \tag{4}$$

and hence, integrating over the solid angle,

$$d\,A=\int\limits_{\theta_1}^{\theta_2}\int\limits_{\gamma_1}^{\gamma_2}\int\limits_{r_0-d}^{r_0+d}\frac{p\,I\,(p)\sin\gamma\,d\gamma\,d\,\theta\,d\,p\,d\,r}{v_r\,(\gamma,\,\theta,\,r,\,\pi\sqrt{2},\,\alpha)}. \tag{5}$$

For θ the limits of integration of (5) are evidently 0 and 2π. Obviously also $\gamma_1=0$. Because the solid angle used is defined by means of shutters placed somewhere between the sample and the detector, γ_2 will depend upon the shape of the orbits, that is, on the field-shape or α. In (3) we supposed that every value of r within the detector slit could be reached by the bundle. However, for paths whose starting directions are near the edge of the solid angle used, this will not be the case. We have to write

$$\left.\begin{aligned} -d &\rightarrow d_1(\gamma, \theta, \alpha), \\ d &\rightarrow d_2(\gamma, \theta, \alpha), \\ \gamma_2 &\rightarrow \gamma_2(\theta, \alpha). \end{aligned}\right\} \tag{6}$$

If (6) is inserted in (5) and the integration carried out over the variables γ, θ and r we get

$$dA = K(\alpha)\, p\, I(p)\, dp, \tag{7}$$

where $K(\alpha)$ for a given spectrometer geometry depends only upon the field parameters $\alpha_1, \alpha_2, \ldots$. The variation of $K(\alpha)$ with the field shape is essentially due to the change in the dispersion and in the solid angle. Below we shall make a quantitative estimate of the variation of $K(\alpha)$, but already now it is perhaps evident that it will normally amount only to a fraction of $K(\alpha)$.

We shall apply (7) to the case that (i) the spectrum consists only of a conversion line and (ii) we have a continuous spectrum.

(i) *Conversion line*

Suppose that $I(p) \neq 0$ only within a narrow interval of p. The area in the $N-H$-plane then becomes

$$A = \int_0^\infty N(H)\, dH = K(\alpha)\, \bar{p} \int_0^\infty I(p)\, dp, \tag{8}$$

where $\bar{p}$ is a mean value of the momentum of the line. By using the calibration function introduced in B we get

$$\frac{\int_0^\infty N(H)\, dH}{H^*} = k(H^*)\, K(\alpha) \int_0^\infty I(p)\, dp \tag{9}$$

if we neglect the finite natural width of the line. We see from (9) that the area under the line divided by its calibration value of H is proportional to the true total conversion intensity within an accuracy given by the constancy of $k(H^*)\, K(\alpha)$.

(ii) *Continuous spectrum*

Let the window curve for electrons of momentum p be given by a function W such that

$$dN(H) = W\left(\alpha, \frac{H}{p}\right) I(p)\, dp, \tag{10}$$

where the dependence of W upon the field shape has been allowed for by letting W contain $\alpha_1, \alpha_2, \ldots$. For the counting rate at H we can write

$$N(H) = I(\bar{p}) \int_0^\infty W\left(\alpha, \frac{H}{p}\right) d\,p, \tag{11}$$

where $\bar{p}$ is a mean value of p in the interval where $W \neq 0$.

Applying (7) and (10), the area under the window curve can be written

$$\int_0^\infty W\left(\alpha, \frac{H}{p}\right) d\,H = p\,K(\alpha) \tag{12}$$

where, according to the assumption made earlier, the α's are constant over the interval of H for which $W \neq 0$.

By introducing the variable $x = H/p$ in (11) and (12) we obtain

$$N(H) = I(\bar{p}) \int_0^\infty W(\alpha, x) \frac{H}{x^2} d\,x = I(\bar{p}) \frac{H}{\bar{x}^2} \int_0^\infty W(\alpha, x)\, d\,x, \tag{13}$$

where $(\bar{x}^2)^{-1}$ is a mean value of $(x^2)^{-1}$, and

$$\int_0^\infty W\left(\alpha, \frac{H}{p}\right) d\,H = p \int_0^\infty W(\alpha, x)\, d\,x = p\,K(\alpha). \tag{14}$$

We can now write

$$\frac{N(H)}{H} = \frac{K(\alpha)}{\bar{x}^2} I(\bar{p}). \tag{15}$$

In practice we correlate $N(H)$ with $I(p)$, where p is computed by means of the calibration function

$$p = k(H)\,H. \tag{16}$$

If in the calibration function we define H^* as the value of H for the center of gravity of the window curve, p from (16) will be very near to $\bar{p}$ in (11) and $1/\bar{x}^2$ very near to $[k(H)]^2$. The experimental intensity $N(H)$ divided by H is then proportional to the true intensity $I(p)$ within an accuracy given by the constancy of $K(\alpha)\,[k(H)]^2$.

D. A quantitative estimate of the variation of $K(\alpha)$

The magnetic field of a double-focusing spectrometer is described by the z-component of the field in the symmetry plane $z = 0$.

$$H_z(r, 0) = H_0 \left[1 + \alpha_1 \frac{r - r_0}{r_0} + \alpha_2 \left(\frac{r - r_0}{r_0}\right)^2 + \cdots\right]. \tag{17}$$

The correct value of α_1 is $\alpha_1 = -\frac{1}{2}$ while α_2 may have different values between $\alpha_2 = \frac{1}{8}$ and $\alpha_2 = \frac{3}{8}$, depending upon how one wishes to distribute the aberrations.

The description (17) supposes full rotational and mirror symmetry of the magnetic field. We shall assume that these properties are present to a sufficient degree in the actual field and estimate the variation of $K(\alpha)$ when the change of the field shape can thus be described by $d\alpha_1$ and $d\alpha_2$. It is evidently changes of this kind which we have to expect primarily.

The reference point for measuring the magnetic field is supposed to be on the central line $r=r_0$, $z=0$. The variation in the calibration function $K(H)$ is then negligible and for our intensity measurements we need only know the variation of $K(\alpha)$.

The expression for $K(\alpha)$,

$$K(\alpha)=\int\limits_{\theta_1}^{\theta_2}\int\limits_{\gamma_1}^{\gamma_2}\int\limits_{r_0+d_1}^{r_0+d_2}\frac{\sin\gamma\, d\gamma\, d\theta\, dr}{v_r(\gamma,\theta,r,\pi\sqrt{2},\alpha)},\tag{18}$$

is of the form

$$\frac{\text{solid angle}\times\text{effective detector slit width}}{\text{dispersion}}.$$

We can define two solid angles Ω and ω at the source:

$\Omega=$ the smallest solid angle which contains all the solid angle elements for which the effective detector slit width $\neq 0$.

$\omega=$ the largest solid angle which contains only solid angle elements for which the effective detector slit width is $2d$.

Clearly ω is contained in Ω.

We first need an estimate of

$$\frac{|\Omega|-|\omega|}{|\Omega|}.$$

We shall assume in the following that the angular position φ_r of the radial shutter is not to far from the source, say $\varphi_r<0.8\sim 45^\circ$ whereas the position of the axial shutter will be left arbitrary.

We shall use a relation

$$\sin\gamma\, d\gamma\, d\theta=f(\varphi_s,\alpha)\, ds\tag{19}$$

expressing that, in a shutter plane $\varphi=\varphi_s$ the area ds of a cross-section of a bundle is proportional to and for given α's depends only on the solid angle of the bundle. This means that we consider only values of γ for which the second order terms in γ in equations (38) and (39) below are small compared to the first order terms in γ for $\varphi=\varphi_s$.

Fig. 2 shows the projection of a path in the plane $z=0$. The path just passes (at $z=z_r$) the outer edge of the radial opening between $r=r_0\pm R$ and the inner edge of the detector slit at $r=r_0-d$. This ray has its starting direction on the limit of Ω because no ray passing the plane $\varphi=\varphi_r$ for the radial shutters at $r>r_0+R$, $z=z_r$ can be made to pass through the detector slit. The magnitude of the field is now decreased until we obtain a path that strikes the radial

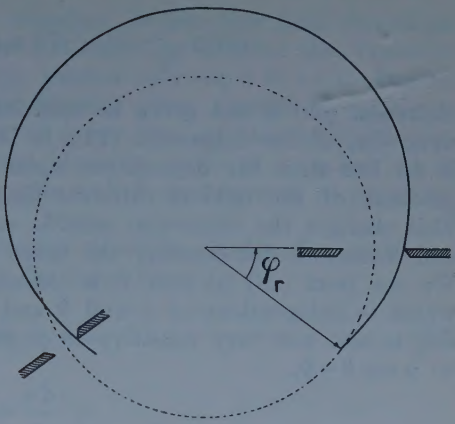

Fig. 2. Projection in the symmetry plane of an orbit with starting direction on the limit of Ω.

shutter at $r = r_0 + R$, $z = z_r$ and the detector slit at its outer edge at $r = r_0 + d$. The starting direction of this ray lies on the limit of ω, because only rays inside it at $\varphi = \varphi_r$, $z = z_r$, can pass through each value of r within the detector slit. In the plane $\varphi = \varphi_r$ the radial difference Δr between the two rays becomes

$$\Delta r = 2\,d\,\frac{v_r(\gamma, \theta, r_0 + R, \varphi_r, \alpha)}{v_r(\gamma, \theta, r_0, \pi\sqrt{2}, \alpha)}. \tag{20}$$

Here we have neglected aberrations, that is, the fact that rays through different points in the plane $\varphi = \varphi_r$ do not make the same intersections in the plane $\varphi = \pi\sqrt{2}$. Using the path equations below, it is easily checked, that this approximation is permissible in this connection.

At the axial opening at $\varphi = \varphi_z$ between $z = \pm Z$ we get correspondingly

$$\Delta z = 2\,d\,\frac{v_z(\gamma, \theta, Z, \varphi_z, \alpha)}{v_r(\gamma, \theta, r_0, \pi\sqrt{2}, \alpha)}, \tag{21}$$

where v_z is the axial dispersion. Considering also the values of Δr and Δz at $r = r_0 - R$ and $z = -Z$ respectively, we get by means of (19)

$$\frac{|\Omega| - |\omega|}{|\Omega|} = 2\,d\left(\frac{v_r(\varphi_r)}{R\,v_r(\pi\sqrt{2})} + \frac{v_z(Z, \varphi_z)}{Z\,v_r(\pi\sqrt{2})}\right) = \varepsilon, \tag{22}$$

where only those dependences of the dispersions which are important here have been indicated. For all generally used values of d/R (22) is of the order only a few percent and its relative variation with the field form is not greater than for the dispersion.

The radial and axial dispersions are proportional to the partial derivatives with respect to δ of equations (39) and (40) resp. below. We need here consider closely only the radial dispersion v_r in the plane $\varphi = \pi\sqrt{2}$. By putting $\varphi = \pi\sqrt{2}$ and $\alpha_1 = -\frac{1}{2} + d\,\alpha_1$ in equation (38) and differentiate with respect to δ we get

$$\text{const}\,\frac{v_r}{r_0}=4\,(1-2\,d\alpha_1)+\frac{\pi\sqrt{2}}{2}\,(1+d\alpha_1)\,\gamma\cos\theta. \tag{23}$$

Equation (39) is not given to sufficiently high order to indicate any dependence upon $d\alpha_2$ of the dispersion (23). It is evident, however, that such a dependence is far less than the dependence upon $d\alpha_1$. Further, a change in α_2 affects the gradient of the field in different directions outside and inside the circle $r=r_0$. This changes the dispersion outside and inside this circle in different directions and hence we can consider the mean variation of v_r as depending on $d\alpha_1$ only. We see from (23) (i) that v_r is essentially determined by the first term in (23), which is independent of γ and θ and (ii) that the relative variation of v_r with $d\alpha_1$ is also not very sensitive to γ and θ. We can take the value corresponding to $\gamma\cos\theta=0$,

$$\frac{d\,v_r}{v_r}=-2\,d\alpha_1 \tag{24}$$

as a good mean value for the relative variation of v_r.

Due to (i) the relative contribution to $K(\alpha)$ from directions between Ω and ω will be only $\sim\frac{1}{2}\varepsilon$. From (i) and (ii) it then follows that the relative variation of $K(\alpha)$ may be written

$$\frac{d\,K(\alpha)}{K(\alpha)}=\frac{d\,|\Omega|}{|\Omega|}-\frac{d\,v_r}{v_r}=\frac{d\,|\omega|}{|\omega|}-\frac{d\,v_r}{v_r}. \tag{25}$$

We shall use $d|\Omega|/|\Omega|$ to find the relative variation of $K(\alpha)$.

Consider the path at the limit of Ω in fig. 2.

Let the radial coordinate of the path be given by a function $r(\gamma,\theta,\varphi,\delta,\alpha_1\,\alpha_2)$. Here the variable δ determines the ratio between p and H. The values of γ, θ and δ are such that

$$r_0+R=r(\gamma,\theta,\varphi_r,\delta,\alpha_1,\alpha_2), \tag{26}$$

$$r_0-d=r(\gamma,\theta,\pi\sqrt{2},\delta,\alpha_1,\alpha_2). \tag{27}$$

Now let the field shape change. With γ and θ constant we compute the radial displacement dr at $\varphi=\varphi_r$ when δ is changed in such a way that the path still just passes the inner edge of the detector slit. Thus

$$d\,r=\frac{\delta r}{\delta\delta}\,d\delta+\frac{\delta r}{\delta\alpha_1}\,d\alpha_1+\frac{\delta r}{\delta\alpha_2}\,d\alpha_2\quad(\varphi=\varphi_r), \tag{28}$$

$$0=\frac{\delta r}{\delta\delta}\,d\delta+\frac{\delta r}{\delta\alpha_1}\,d\alpha_1+\frac{\delta r}{\delta\alpha_2}\,d\alpha_2\quad(\varphi=\pi\sqrt{2}) \tag{29}$$

from which we obtain $$d\,r=s_1(\gamma,\theta,d\alpha_1,d\alpha_2,\varphi_r). \tag{30}$$

If z_r is the axial intersection of the path of equations (26) and (27) with the plane $\varphi=\varphi_r$, γ and θ are unique functions of r_0+R and z_r. Hence we can write

$$d\,r=s_2(r_0+R,z_r,d\alpha_1,d\alpha_2,\varphi_r). \tag{31}$$

Suppose that for the path in fig. 2 dr is negative. In the new field, directions outside Ω can then pass the radial shutter and reach the detector slit. Consider the path which now passes the edge of the shutter at $r=r_0+R$, $z=z_r$. Due to aberrations this ray does not interest the detector plane exactly at $r=r_0-d$. However, with an additional small change of δ, we can get the ray which just passes the radial shutter also to cross the detector plane at $r=r_0-d$. It is found from the path equations below that the radial displacement at $\varphi=\varphi_r$ by this last change of δ is negligible.

Analogously with (31) the axial displacement $dz=t_2(r_z, Z, d\alpha_1, d\alpha_2, \varphi_z)$ in the plane $\varphi=\varphi_z$ is obtained by differentiating the equation

$$z=z(\gamma, \theta, \varphi, \delta, \alpha_1, \alpha_2), \tag{32}$$

where γ, θ and δ satisfy

$$Z=z(\gamma, \theta, \varphi_z, \delta, \alpha_1, \alpha_2), \tag{33}$$

$$r_0-d=r(\gamma, \theta, \pi\sqrt{2}, \delta, \alpha_1, \alpha_2). \tag{34}$$

Let the beam containing all rays which can pass through the detector slit be extended axially between appr. $-Z_r$ and Z_r in the plane of the radial opening and extended radially between appr. r_0-R_z and r_0+R_z in the plane of the axial opening. By means of relation (19) we may write the relative change of $|\Omega|$

$$\frac{d|\Omega|}{|\Omega|}=\frac{\int\limits_{-Z_r}^{Z_r}[s_2(r_0-R, z_r, d\alpha_1, d\alpha_2, \varphi_r)-s_2(r_0+R, z_r, d\alpha_1, d\alpha_2, \varphi_r)]\,dz_r}{4RZ_r}+$$
$$+\frac{\int\limits_{r_0-R_z}^{r_0+Rz}[t_2(r_z, -Z, d\alpha_1, d\alpha_2, \varphi_z)-t_2(r_z, Z, d\alpha_1, d\alpha_2, \varphi_z)]\,dr_z}{4ZR_z}. \tag{35}$$

With the functions r and z given below, we compute approximately s_2 and t_2. Finally, we find (for small φ_r as assumed above) by means of (24) and (25)

$$\frac{dK(\alpha)}{K(\alpha)}=\left(1-0{,}5\,\varphi_r+\frac{\varphi_z}{\sqrt{2}}\cot\frac{\varphi_z}{\sqrt{2}}\right)d\alpha_1. \tag{36}$$

For $\varphi_z=\varphi_r=\pi[\tfrac{1}{2}(\sqrt{2}-1)]=37^\circ$ which is often used, we get

$$\frac{dK(\alpha)}{K(\alpha)}=1.7\,d\alpha_1 \tag{37}$$

while for $\varphi_z=157^\circ$, $\varphi_r=37^\circ$ $\qquad \dfrac{dK(\alpha)}{K(\alpha)}=0. \qquad (38)$

The functions $r(\gamma, \theta, \varphi, \delta, \alpha_1, \alpha_2)$ *and* $z(\gamma, \theta, \varphi, \delta, \alpha_1, \alpha_2)$

Let
$$\varrho = \frac{r - r_0}{r_0}, \qquad \zeta = \frac{z}{r_0}.$$

From Svartholm [2] we have, according to (7: 9) and (7: 10),

$$\varrho^{(0)} = \frac{\gamma}{\sqrt{\lambda_r}} \cos\theta \sin\varphi\sqrt{\lambda_r},$$

$$\zeta^{(0)} = \frac{\gamma}{\sqrt{\lambda_z}} \sin\theta \sin\varphi\sqrt{\lambda_z}.$$

These functions are the first order solutions for ϱ and ζ in the case where the momentum p_0 of the particles is such that the circle $r = r_0$, $z = 0$ is a possible trajectory. The second order solutions under the same condition are according to (7: 13), (7: 14) and (7: 15)

$$\varrho = \varrho^{(0)} + \varrho^{(1)}, \qquad \zeta = \zeta^{(0)} + \zeta^{(1)},$$

where

$$\begin{aligned}\varrho^{(1)} = {} & \frac{1}{\lambda_r}\left[\frac{f_{11}}{\lambda_r}\cos^2\theta + \left(\varkappa - 2 + \frac{f_{22}}{\lambda_z}\right)\sin^2\theta\right]\tfrac{1}{2}\gamma^2\left(1 - \cos\varphi\sqrt{\lambda_r}\right) + \\ & + \frac{1}{3\,\lambda_r}\left(\varkappa - 2 + \frac{f_{11}}{\lambda_r}\right)\tfrac{1}{2}\gamma^2\cos^2\theta\left(\cos 2\varphi\sqrt{\lambda_r} - \cos\varphi\sqrt{\lambda_r}\right) + \\ & + \frac{1}{4\,\lambda_z - \lambda_r}\,\frac{f_{22}}{\lambda_z}\,\tfrac{1}{2}\gamma^2\sin^2\theta\left(\cos 2\varphi\sqrt{\lambda_z} - \cos\varphi\sqrt{\lambda_r}\right),\end{aligned}$$

$$\begin{aligned}\zeta^{(1)} = {} & \frac{1}{(\sqrt{\lambda_r} - \sqrt{\lambda_z})^2 - \lambda_z}\left(\varkappa - 2 - \frac{g_{12}}{\sqrt{\lambda_r\,\lambda_z}}\right)\tfrac{1}{2}\gamma^2\sin\theta\cos\theta\left[\cos\varphi\left(\sqrt{\lambda_r} - \sqrt{\lambda_z}\right) - \right. \\ & \left. - \cos\varphi\sqrt{\lambda_z}\right] + \frac{1}{(\sqrt{\lambda_r} + \sqrt{\lambda_z})^2 - \lambda_z}\left(\varkappa - 2 + \frac{g_{12}}{\sqrt{\lambda_r\,\lambda_z}}\right)\cdot \\ & \cdot\tfrac{1}{2}\gamma^2\sin\theta\cos\theta\left[\cos\varphi\left(\sqrt{\lambda_r} + \sqrt{\lambda_z}\right) - \cos\varphi\sqrt{\lambda_z}\right].\end{aligned}$$

For our case, i.e. the pure magnetic one, we have $\varkappa = 1$ and

$$\lambda_r = 1 + \alpha_1,$$

$$\lambda_z = -\alpha_1,$$

$$f_{11} = -\tfrac{1}{2} - \tfrac{3}{2}\alpha_1 - \alpha_2,$$

$$f_{22} = \alpha_2,$$

$$g_{12} = 2\,\alpha_1 + 2\,\alpha_2.$$

When the momentum p differs by $\delta \cdot p = p - p_0$ from the value above we have, according to (6:6) and (6:7),

$$\varrho^* = \frac{\gamma}{\sqrt{\bar{\lambda}_r}} \cos\theta \sin\varphi \sqrt{\bar{\lambda}_r} + \frac{\delta}{\bar{\lambda}_r} \frac{2 - \varkappa + \delta}{(1+\delta)^2} (1 - \cos\varphi \sqrt{\bar{\lambda}_r}),$$

$$\zeta^* = \frac{\gamma}{\sqrt{\bar{\lambda}_z}} \sin\theta \sin\varphi \sqrt{\bar{\lambda}_z},$$

where for $\varkappa = 1$ $\quad \bar{\lambda}_r = \lambda_r - \frac{\delta}{2}, \quad \bar{\lambda}_z = \lambda_z - \frac{\delta}{2}.$

By using ϱ^*, $\varrho^{(1)}$, ζ^*, and $\zeta^{(1)}$ we obtain for the functions r and z

$$r = r_0 (1 + \varrho^* + \varrho^{(1)}), \tag{39}$$

$$z = r_0 (\zeta^* + \zeta^{(1)}). \tag{40}$$

For small values of δ a change $\delta \cdot p$ in the momentum is evidently equivalent to a change $-\delta \cdot H$ of the measured magnetic field strength. Hence, the dispersions are proportional to the partial derivatives with respect to δ of (39) and (40).

We can here discuss the assumption of regarding the field parameters α_i as constants over the interval of H which is contained by a line profile. First we observe that, for a continuous spectrum, we measure the counting rate $N(H)$ at a certain fixed field setting H. The counting rate $N(H)$ can evidently depend only on the values of α_i for that particular field strength, whereas the derivatives of α_i with respect to H will not enter. Hence, for a continuous spectrum the "static approximation" $d\alpha_i/dH = 0$ is exact. When measuring a conversion line, however, the relevant entity in (9) is the area $\int_0^\infty N\,dH$ under the line profile. Because the measurement of line intensities thus involves a variation of H, the derivatives of α_i must in principle also enter into a relation between the measured area and the true conversion intensity. The dependence on the derivatives of α_i is introduced mainly with their influence on the dispersion v_r in the focal plane. If $d\alpha_i/dH \neq 0$ the partial derivative of (39) with respect to δ will contain $d\alpha_i/d\delta$. However, an appreciable change of the dispersion due to these terms requires extremely large values of $d\alpha_i/d\delta$, which are most unlikely to occur.

E. Application to the two-directional focusing spectrometer with $r_0 = 50$ cm at the Nobel Institute of Physics

The double-focusing spectrometer at the Nobel Institute (3) has a magnet of which the pole pieces are connected by a central core, around which the magnetizing coils are wound. The magnetic field is measured by means of two rotating coils, mounted on the same axis. One coil has its center on the central line $r = r_0$, $z = 0$ at $\varphi = \pi [\frac{1}{2}(\sqrt{2} + 2)]$, while the other is situated in an iron-free coil of Helmholtz type, the field of which is known. The phase difference between the voltages induced in the two coils is 180° and the current in the spectrometer coils is varied until a minimum in the sum of the voltages from the rotating coils is observed. Thus the mean field strength around the measuring point in the spectrometer field is proportional to the field strength in the Helmholtz coil.

1. *The energy dependence of the shape of the spectrometer field*

The change of the field shape was investigated in the following way. A shutter with three openings was placed at $\varphi = \pi (\frac{1}{2}\sqrt{2})$. The openings had their centres in the symmetry plane at $r = 40$, 50 and 60 cm. For a given conversion energy we obtain three lines; by studying how their relative positions vary with the energy, we can compute the corresponding values of $d\alpha_1$ and $d\alpha_2$. Fig. 3 shows the positions in the $N-H$-plane of the lines from the openings at $r = 40$ cm and $r = 60$ cm relative to the position of the line corresponding to the central opening for four different electron energies. All measurements involving the field form were made with the magnet demagnetized except where otherwise indicated. Under these conditions the field shape is sufficiently reproducible.

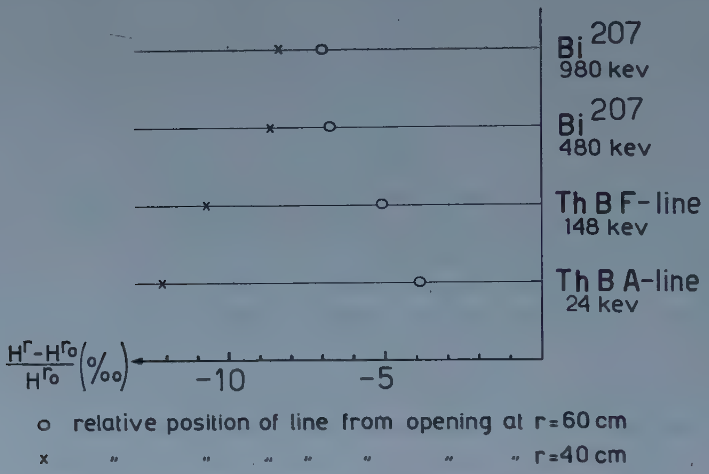

Fig. 3. Diagram showing the relative position in H of the lines from the openings in a shutter at $\varphi = \pi (\frac{1}{2}\sqrt{2})$.

For the lines from the openings at $r = r_0 = 50$ cm, one finds that the field strength corresponding to the peaks of the lines is proportional to the $B\varrho$-values of the conversion lines within ~ 1 ‰ for the investigated energy interval 24–1000 keV. At $r = 35$ and $r = 65$ cm where the extreme rays pass at maximum transmission of the spectrometer, we find from fig. 3 that between 24 and 1000 keV the relative field strength at these radii has changed with 7 ‰ and -7 ‰ respectively. An investigation of the axial focusing showed that no appreciable asymmetric changes occurred. It is therefore evident that the change of the field shape has essentially the character, which is described by means of $d\alpha_1$ and $d\alpha_2$. We shall discuss briefly how small changes of other kinds will affect $K(\alpha)$. Changes of the following kind may be present:

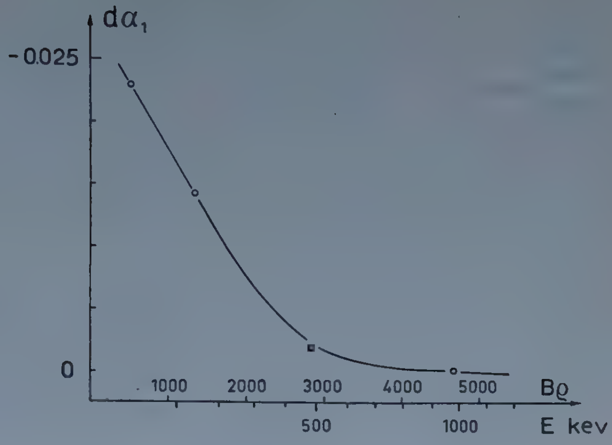

Fig. 4. Diagram showing the variation of α_1 with the energy of the focused electrons.

(a) Changes which are not symmetric around the plane $z=0$.
(b) The change of the field gradient is not independent of φ.
(c) The field strength along the central line $r=r_0$, $z=0$ does not remain independent of φ.

We first observe that the change of $K(\alpha)$ is determined by the character of the change of the field only, i.e. it is insensitive to small differences in the shape of the field from which we start. Because of this, we can consider (a), (b) and (c) separately; in reality there are, of course, combinations of these. We suppose, therefore, that we start from a field with rotational and mirror symmetry and change it according to one of (a), (b) and (c) with the condition that the values of $d\alpha_1$, $d\alpha_2$ that we measure experimentally with the method above are kept constant. A change of type (a) will affect both the dispersion and the solid angle in different directions on different sides of the plane $z=0$ and hence $K(\alpha)$ is insensitive to small changes of this type. If the field shape changes according to (b), our measurement gives a mean value of the gradient, which is clearly of significance for the value of $K(\alpha)$. Effects from changes under (c) enter mainly in the functions s_2 in (35). However, the small changes introduced will be almost the same for $s_2(r+R)$ and $s_2(r-R)$ and hence cancel in expression (35) for $d|\Omega|/|\Omega|$.

Also in the real case, the relation (36) therefore gives the intensity correction to be used, provided that $d\alpha_1$ is determined as above. Fig. 4 shows $d\alpha_1$ obtained from the measurements of fig. 3 as a function of the focused energy and where $d\alpha_1$ has arbitrarily been put equal to zero at 1000 keV. The values of $d\alpha_2$ compatible with fig. 3 are sufficiently small to justify the approximation involved by the non-appearance of $d\alpha_2$ in relation (36).

2. *Experimental check of the intensity correction*

The most straightforward way to check experimentally the computed intensity variation would, of course, be to measure a radiation with known intensity distribution over the energy interval used. However, concerning conversion lines,

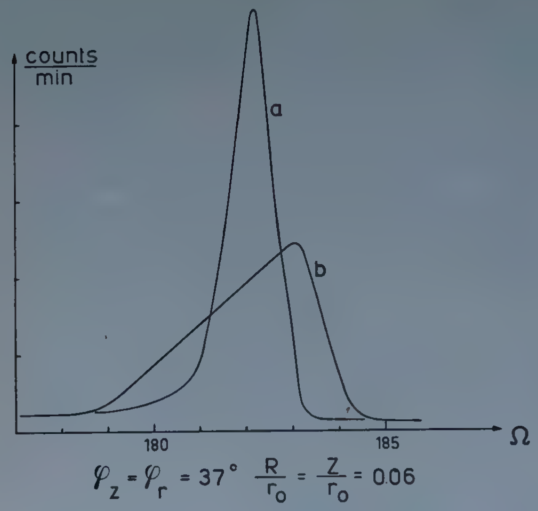

Fig. 5. Line shapes of ThB *A*-line, (*a*) with the spectrometer demagnetized, (*b*) with the spectrometer premagnetized with 6 amps.

intensity ratios are rarely known with an accuracy of 1 % or better which we require for our purpose. On the other hand, a continuous β-spectrum of a known shape is also difficult to use because of back scattering in the sample, especially at low energies. It may also be difficult to obtain sufficient statistical accuracy at all energies.

The difficulties mentioned are avoided if we measure at constant energy and vary instead the field shape by giving the spectrometer magnet a certain premagnetization.

It is most suitable to use a continuous spectrum to test the intensity variation. The experimental errors are then easier to estimate because they are essentially made up only of the statistical uncertainty in the counting rate. The measurement of a line introduces further uncertainty in the field settings for the measuring points and in the determination of the area under the line. When using a continuous spectrum, we also test just the static approximation which we have assumed in the computation. The results then also hold for conversion lines, provided that there are no rapid changes in the field shape. This is evidently more a question of reproducibility than a systematic intensity variation with the field shape. The difficulty is easily eliminated by measuring a line with fixed field setting and varying the energy of the electrons by means of an accelerator at the sample. However, no discrepancies were observed for some measurements of lines performed in the ordinary way.

To get a sufficiently large change in the field shape when premagnetizing the spectrometer, the measurements had to be made at rather low energies. Fig. 5 shows for one particular shutter setting the line shape of the *A*-line of ThB at 24.5 keV (*a*) with the spectrometer demagnetized and (*b*) with a premagnetization of 6.0 amp. The focusing current for the *A*-line is ~ 0.2 amp. The variable Ω in fig. 5 is the setting of the potentiometer in a bridge coupling and is proportional to the current through the Helmholtz coil. *H* and Ω are related by $H = \text{const}\ (\Omega + 7.8)$ where the term 7.8 is due to the earth magnetic field. Al-

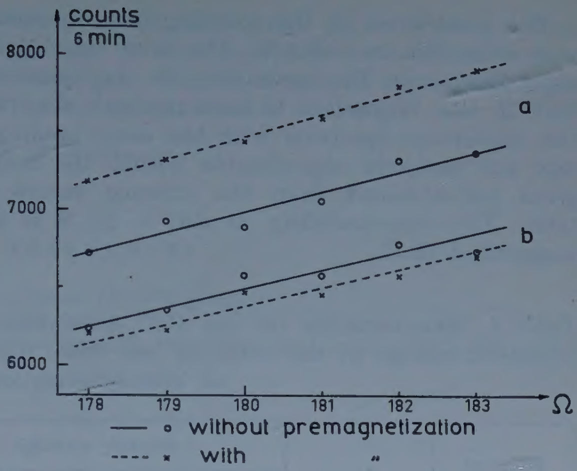

Fig. 6. Results from measurements on the Co^{60} continuum at ~24 keV.
(a) $\varphi_z = \varphi_r = 37°$,
(b) $\varphi_z = 157°$, $\varphi_r = 37°$.

though the line shape is for several reasons not perfectly identical with the window curve, fig. 5 illustrates approximately the change of the window curve when the spectrometer is premagnetized.

Measurements with the three-slits shutter at $\varphi = \pi(\frac{1}{2}\sqrt{2})$ gave values for $d\alpha_1$ around 0.043, whereas $d\alpha_2$ was too small to be determined. It was checked that no large changes of the field form of the types under (*a*), (*b*) and (*c*) above were introduced with the premagnetization.

The constancy of the G.M. tube and the counter arrangement was checked at regular intervals by means of a gamma-source which could be placed in a well defined position outside the spectrometer.

The vacuum in the spectrometer was held sufficiently low and sufficiently constant to prevent any appreciable variation in the counting rate.

The counting rate has some contribution from electrons scattered in the spectrometer. The intensity variation of these electrons may not follow the relation (36) which we wish to test. However, an investigation of the axial intensity distribution in the detector plane showed that the contribution to the counting rate from unfocused electrons was not greater than 1 % and hence cannot affect the variation of N with the field shape.

Fig. 6 shows measurements on the Co^{60} continuum. For each measuring point the statistical error in the counting rate is slightly above 1 %. For (*a*) both the radial and the axial shutters are placed at their normal positions at $\varphi = 37°$, while for (*b*) the axial shutter has been moved to $\varphi = 157°$. From measurements on the A-line it was found that for the shutter settings used the Ω-value for the centre of gravity of the window curve remained nearly constant when the spectrometer was premagnetized. Fig. 5 illustrates this fact for one of the shutter settings. Therefore, in spite of the rather steep slope of the part of the continuous spectrum in fig. 6, no appreciable error is introduced if we determine the change of $K(\alpha)$ by comparing counting rates with the same value of Ω.

Table 1 contains shutter settings used and the computed and obtained results. The detector slit width used was 0.5 cm. Figures from measurements around the energy of the F-line at 148 keV are also given.

The mean error in the counting rate, obtained from several points as in fig. 6, was estimated to ~0.5 %. The error in dN/N given in Table 1 is taken as twice this error. The uncertainty in $d\alpha_1$ indicated in Table 1 is due to the fact that it was impossible to measure both $d\alpha_1$ (with the three-slits shutter) and the continuous spectrum with the same premagnetized field. The change of α_1 was not perfectly reproducible within the measuring accuracy and the errors given are obtained from the extreme values of $d\alpha_1$ from several magnetizations. The reproducibility of $d\alpha_1$ in fig. 4 is at least as good as for the premagnetized field.

Table 1. Measurements on the Co^{60} continuum to compare the observed and computed change of the counting rate when the field shape is changed by means of premagnetization.

Focused energy keV	$d\alpha_1$ measured	Shutter settings			$\frac{dK(\alpha)}{K(\alpha)}$ % theoretical	$\frac{dN}{N}$ % measured
		$\frac{R}{r_0}$	$\frac{Z}{r_0}$	φ_z		
24	0.043 ± 0.005	0.06	0.06	37°	7.3 ± 0.9	7.1 ± 1.0
		0.06	0.10	157°	0	-1.7 ± 1.0
		0.10	0.10	37°	7.3 ± 0.9	7.3 ± 1.0
		0.10	0.20	157°	0	-1.3 ± 1.0
140	0.020 ± 0.003	0.05	0.05	37°	3.4 ± 0.5	3.0 ± 1.0
		0.05	0.08	157°	0	0.6 ± 1.0

As is seen from Table 1, the experimental values are in reasonable agreement with the predicted variation. If we choose $\varphi_z = \varphi_r = 37^\circ$, and apply the values of $d\alpha_1$ for the normal variation of the field form with focused energy from fig. 4, the error in relative intensities due to this cause should not exceed ~2 %. For $\varphi_z = 157^\circ$ the uncertainty should be kept within 2 % without any correction for $d\alpha_1$.

SUMMARY

Considering a two-directional focusing β-spectrometer with the opening radially defined in the plane $\varphi = \varphi_r$ and axially in the plane $\varphi = \varphi_z$ an investigation of how the measured intensities are affected by a change of the field shape of the spectrometer is made. The change of the field shape is assumed to be a smooth function of the focused energy. In section C a general expression is obtained which indicates that both the intensity of a conversion line (if defined as the area under the line profile divided by the $B\varrho$ value) and the intensity of a continuous spectrum will normally vary only slightly with a change of the field shape. In section D a quantitative estimate of the variation is made for the case that the field change possesses mirror and rotational symmetry and the axial and radial shutters are set symmetrically with respect to the central line. With φ_r fixed at 37°, it is found that for $\varphi_z = 157°$ the intensity variation vanishes, while for $\varphi_z = 37°$ its relative value is approximately equal to the relative change of the field gradient, but with opposite sign. In the following sections the results are applied to the two-directional focusing spectrometer at the Nobel Institute. Effects of small non-symmetrical field changes are discussed. Reasonably good agreement between the results from section D and the experimental values is obtained.

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